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Solution of a Hamiltonian of quantum dots with Rashba spin–orbit coupling: quasi-exact solution

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Abstract

We present a method to solve the problem of Rashba spin–orbit coupling in semiconductor quantum dots, within the context of quasi-exactly solvable spectral problems. We show that the problem possesses a hidden $osp(2, 2)$ superalgebra. We constructed a general matrix whose determinant provides exact eigenvalues. Analogous mathematical structures between the Rashba and some of the other spin-boson physical systems are noted.

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1. Introduction

The optical and electrical properties of confined electrons in semiconductor quantum wells, quantum dots and quantum wires depend on Rashba spin–orbit coupling [1–3]. The analysis of the behaviour of spins in semiconductors leads to the construction of new *spintronic* devices [4–7]. It is practical to contemplate semiconductor devices based on electron spins, because spin is not coupled to electromagnetic noise and hence should have much longer coherence times than charge [8]. The spin–orbit interaction in a confined geometry has also interesting consequences for the electron spectrum [9].

Spin–orbit interactions can arise in quantum dots by various mechanisms related to the electron confinement and symmetry breaking. In semiconductors, spin–orbit coupling from the relativistic effect is caused by the electric field due to the lack of the inversion symmetry of the certain alloys. Depending on the particular origin of the electric field, the spin–orbit interaction presents two distinct contributions: these are Dresselhaus and Rashba terms. Dresselhaus term is obtained from the electric field produced by the bulk inversion asymmetry of the material and Rashba term is generated due to the structural asymmetry of the heterostructure. Rashba splitting has been observed in many experiments and it constitutes the basis of the proposed electronic nanostructures.

In the literature, most of the studies about the solution of the spin–orbit effects in quantum dots are carried out by means of perturbation theory or numerical methods, because the implementation of the algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. The aim of this paper is to provide exact solution of a quantum dot Hamiltonian including Rashba and Zeeman terms by an algebraic formulation. Analytical solution of the problem has recently been treated by employing various techniques [9–13]. We demonstrate that the exact eigenfunctions and eigenvalues are available for the Hamiltonian of a quantum dot including Rashba coupling in the framework of quasi-exact solvability (QES) [14–17]. As a related topic the concept of quasi-exactly solvable systems discovered in the 1980s has received much attention in recent years, both from the viewpoint of physical applications and their mathematical beauty. It turns out that in quantum mechanics there exist such systems; a part of their spectrum can be computed using algebraic methods. We also show that the corresponding Hamiltonian possesses a hidden $osp(2, 2)$ algebraic structure.

Our formulation of the Hamiltonian associated with the quantum dots leads to an interesting consequence: Hamiltonians of the $E \otimes \varepsilon$ Jahn–Teller [18–22], two-mode bosonic Jaynes–Cummings [23–25] and quantum dot Hamiltonians with spin–orbit interaction are constructed for similar spin-boson models, though not identical, and can be solved with the same mathematical techniques. The connection between the Hamiltonians plays crucial roles to the solution and analysis of the Rashba Hamiltonian, because the solutions of the Jahn–Teller and the Jaynes–Cummings problems have been a focus of interest, both from the mathematical and physical point of view, over the years and their exact solution has been treated by various authors.

The paper is organized as follows. In section 2, we briefly review the construction of the Hamiltonian that includes Rashba spin–orbit coupling term. Section 3 is devoted to algebrization of the corresponding Hamiltonian. In this section, we also discuss the symmetry properties of the Hamiltonian. In section 4, we present a transformation procedure that is appropriate to determine the QES of the Hamiltonian. Finally, we conclude our results in section 4.

2. The model Hamiltonian

The origin of Rashba spin–orbit coupling in quantum dots is due to the lack of inversion symmetry which causes a local electric field perpendicular to the plane of heterostructure. The Hamiltonian representing Rashba spin–orbit coupling for an electron in a quantum dot can be expressed as

$$H_R = \frac{\lambda_R}{\hbar}(p_y\sigma_x - p_x\sigma_y) \tag{1}$$

where λ_R represents the strength of the spin–orbit coupling, and it can be adjusted by changing the asymmetry of the quantum well via external electric field and the matrices σ_x and σ_y are Pauli matrices. The Hamiltonian dominated by the bulk inversion symmetry term is known as Dresselhaus spin–orbit coupling and it is given by

$$H_D = \frac{\lambda_D}{\hbar}(p_y\sigma_y - p_x\sigma_x) \tag{2}$$

where λ_D is the Dresselhaus parameter that depends on the material properties, device design and external electric field. In a centrosymmetric crystal-like materials it becomes zero, because of the non-existence of bulk inversion asymmetry. In the case of both Rashba and Dresselhaus, interactions exist in the system, the Hamiltonian cannot exactly be solved. In our treatment we

have included the Rashba term rather than the Dresselhaus term, because the Rashba term may be dominant, since the typical ratio of the coupling strengths $\lambda_R/\lambda_D = 6$ [11]. We mention here that the results will only need a trivial modification, when a solo Dresselhaus term is present, because Rashba and Dresselhaus terms transform into each other under spin rotation. To this end, we assume that the electron is confined in a parabolic potential

$$V = \tfrac{1}{2}m^*\omega_0^2(x^2 + y^2) \tag{3}$$

here m^* is the effective mass of the electron and ω_0 is the confining potential frequency. The Hamiltonian describing an electron in a two-dimensional quantum dot takes the form

$$H = \frac{1}{2m^*}\left(P_x^2 + P_y^2\right) + \frac{1}{2}g\mu B\sigma_z + V + H_R. \tag{4}$$

The term $\frac{1}{2}g\mu B\sigma_z$ introduces the Zeeman splitting between the $(+)x$-polarized spin up and $(-)x$-polarized spin down. The factors g is the gyromagnetic ratio and μ is the Bohr magnetron. The kinetic momentum $\mathbf{P} = \mathbf{p} + \mathbf{eA}$ is expressed with canonical momentum $\mathbf{p} = -\mathrm{i}\hbar(\partial_x, \partial_y, 0)$ and the vector potential $\mathbf{A}$ can be related with the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The choice of symmetric gauge vector potential $\mathbf{A} = B/2(-y, x, 0)$ leads to the following Hamiltonian:

$$H = -\frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}m^*\omega^2(x^2 + y^2) + \frac{1}{2}\mathrm{i}\hbar\omega_c\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \frac{1}{2}g\mu B\sigma_z + H_R \tag{5}$$

where $\omega_c = eB/m^*$ stands for the cyclotron frequency of the electron, $\omega = \sqrt{\omega_0^2 + \left(\frac{\omega_c}{2}\right)^2}$ is the effective frequency. From now on we restrict ourselves to the solution of (5). It will be shown that the Hamiltonian (5) without the Dresselhaus term is one of the recently discovered quasi-exactly solvable operators [14, 15]. It is well known that the underlying idea behind the quasi-exact solvability is the existence of a hidden algebraic structure. In the following section, we obtain an algebraic expression for the Hamiltonian H and discuss its quasi-exact solvability.

3. QES of the Hamiltonian

The general procedure to solve a differential equation quasi-exactly is to express it in terms of the given Lie algebra having a finite-dimensional invariant subspace and use algebraic operations. One way to relate the Hamiltonian H with an appropriate Lie algebra is to construct its bosonic and fermionic representation. For this purpose, let us introduce the following bosonic operators:

$$\begin{aligned}
a_1^+ &= \sqrt{\frac{m^*\omega}{4\hbar}}(x + \mathrm{i}y) - \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x + \mathrm{i}\partial_y) \\
a_1 &= \sqrt{\frac{m^*\omega}{4\hbar}}(x - \mathrm{i}y) + \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x - \mathrm{i}\partial_y) \\
a_2^+ &= \sqrt{\frac{m^*\omega}{4\hbar}}(x - \mathrm{i}y) - \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x - \mathrm{i}\partial_y) \\
a_2 &= \sqrt{\frac{m^*\omega}{4\hbar}}(x + \mathrm{i}y) + \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x + \mathrm{i}\partial_y).
\end{aligned} \tag{6}$$

They satisfy the usual commutation relations. The Hamiltonian H describing a two-level fermionic subsystem coupled to two-boson modes can be expressed as

$$H = \hbar\omega\left(a_1^+a_1 + a_2^+a_2 + 1\right) + \frac{\hbar\omega_c}{2}\left(a_1^+a_1 - a_2^+a_2\right) - \sqrt{\frac{m^*\omega}{\hbar}}\lambda_R\left[\left(a_2^+ - a_1\right)\sigma_+ + \left(a_2 - a_1^+\right)\sigma_-\right] + \frac{1}{2}g\mu B\sigma_0. \tag{7}$$

The Pauli matrices are given by

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Before we turn to exploring the solvability of the Hamiltonian H we briefly discuss its Lie (super)algebraic properties.

The natural step to relate a Hamiltonian with a Lie (super)algebra is to express the Hamiltonian with the generators of the relevant symmetry group. One of the major symmetry group candidates for spin one-half particles is the supergroup $osp(2, 2)$ which has four even and four odd generators. Its even generators can be represented by bosons while odd generators are represented by combinations of the fermions and bosons [26, 27]. The Lie superalgebra $osp(2, 2)$ can be constructed by extending $su(1, 1)$ algebra whose generators are given by

$$J_+ = a_1^+a_2^+, \qquad J_- = a_2a_1, \qquad J_0 = \tfrac{1}{2}\left(a_1^+a_1 + a_2^+a_2 + 1\right). \tag{9}$$

These are the Schwinger representation of $su(1, 1)$ algebra and its number operator is given by

$$N = a_1^+a_1 - a_2^+a_2 \tag{10}$$

which commutes with the $su(1, 1)$ generators. The superalgebra $osp(2, 2)$ might be constructed by extending $su(1, 1)$ algebra with the fermionic generators

$$V_+ = \sigma_+a_2^+, \qquad V_- = \sigma_+a_1, \qquad W_+ = \sigma_-a_1^+, \qquad W_- = \sigma_-a_2. \tag{11}$$

The total number operator of the system is given by

$$J = \tfrac{1}{2}(N - \sigma_0). \tag{12}$$

The conserved quantity of a physical system possesses $osp(2, 2)$ superalgebra can be written as

$$K = N - \tfrac{1}{2}\sigma_0 \tag{13}$$

which commutes with the operators of the $osp(2, 2)$ superalgebra. The generators of the $osp(2, 2)$ superalgebra satisfy the following commutation and anti-commutation relations:

$$\begin{aligned}
&[J_+, J_-] = -2J_0, && [J_0, J_\pm] = \pm J_\pm, \\
&[J, J_\pm] = 0, && [J, J_0] = 0 \\
&[J_0, V_\pm] = \pm\tfrac{1}{2}V_\pm, && [J_0, W_\pm] = \pm\tfrac{1}{2}W_\pm, \\
&[J_\pm, V_\mp] = V_\pm, && [J_\pm, W_\mp] = W_\pm, \\
&[J, W_\pm] = -\tfrac{1}{2}W_\pm, && [J, V_\pm] = \tfrac{1}{2}V_\pm \\
&[J_\pm, V_\pm] = 0, && [J_\pm, W_\pm] = 0 \\
&\{V_\pm, W_\pm\} = J_\pm, && \{V_\pm, W_\mp\} = \pm J_0 - J \\
&\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = 0 \\
&\{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0.
\end{aligned} \tag{14}$$

The Hamiltonian of a physical system, with an underlying $osp(2, 2)$ symmetry, has been expressed in terms of the operators of the corresponding algebra. In general, the Hamiltonian is

exactly solved and the spectrum of the physical system can be calculated in a closed form when the Hamiltonian of the system can be written in terms of number operator and/or diagonal operator J_0, or it can be diagonalized within the representation $[J]$. The abstract boson and/or fermion algebra can be associated with the exactly solvable Schrödinger equations by using the differential operator realizations of boson operators. This connection opens the way to an algebraic treatment of a large class of potentials of practical interest [28]. The combinations of the operators of $osp(2,2)$ superalgebra have direct physical meaning, being related to the quantum spin systems.

The Hamiltonian (7) can be expressed in terms of operators of $osp(2,2)$ algebra

$$H = 2\hbar\omega J_0 + \frac{\hbar\omega_c}{2}N - \kappa\left[V_+ - V_- + W_- - W_+\right] + g\mu B\left(N - K\right), \tag{15}$$

where $\kappa = \sqrt{\frac{m^*\omega}{\hbar}}\lambda_R$. Consequently, we have shown that the Rashba Hamiltonian possesses the $osp(2,2)$ symmetry. The Hamiltonian (15) has $(2N+1)$ linearly independent eigenfunctions, and it is quasi-exactly solvable.

Now we turn our attention to the solution of the Hamiltonian (7). A simple connection between the Hilbert space and the Bargmann–Fock space can be obtained by transforming the differential realizations of the creation and annihilation operators (6). This can be done by introducing the following similarity transformation operators:

$$\Lambda = \exp\left[-\frac{\pi}{4}\left(a_1^+a_2 + a_2^+a_1\right)\right]; \qquad \Gamma = \exp\left[\frac{\pi}{8}\left(a_1^2 + a_1^{+2} - a_2^2 - a_2^{+2}\right)\right]. \tag{16}$$

The similarity transformation and change of the variable $y \to \mathrm{i}y$ give the following realizations:

$$\begin{aligned} b_1 &= \Gamma\Lambda a_1\Lambda^{-1}\Gamma^{-1} = \sqrt{\frac{\hbar}{m^*\omega}}\frac{\partial}{\partial x}, \qquad & b_1^+ &= \Gamma\Lambda a_1^+\Lambda^{-1}\Gamma^{-1} = \sqrt{\frac{m^*\omega}{\hbar}}x \\ b_2 &= \Gamma\Lambda a_2\Lambda^{-1}\Gamma^{-1} = \sqrt{\frac{\hbar}{m^*\omega}}\frac{\partial}{\partial y}, \qquad & b_2^+ &= \Gamma\Lambda a_2^+\Lambda^{-1}\Gamma^{-1} = \sqrt{\frac{m^*\omega}{\hbar}}y. \end{aligned} \tag{17}$$

It is obvious that the operator K commutes with the whole Hamiltonian and the eigenvalue problem

$$K|n_{1,}n_2\rangle = \left(k + \tfrac{1}{2}\right)|n_{1,}n_2\rangle \tag{18}$$

in the Bargmann–Fock space leads to the following solution:

$$\psi(x,y) = x^k\phi(xy)|\uparrow\rangle + x^{k+1}\phi(xy)|\downarrow\rangle, \tag{19}$$

where $|\uparrow\rangle$ stands for up state and $|\downarrow\rangle$ stands for down state. The eigenfunction of the Hamiltonian can be obtained from the relation

$$|n_1,n_2\rangle = \Lambda^{-1}\Gamma^{-1}\psi(x,y). \tag{20}$$

Substitution of (19) into the Hamiltonian (7) leads to the following set of one variable differential equations:

$$\begin{aligned} &\hbar\omega\left[2z\frac{\mathrm{d}}{\mathrm{d}z} + k + 1 + \frac{k\omega_c}{2\omega} - \frac{\mu gB}{2\hbar\omega} - \frac{E}{\hbar\omega}\right]\phi_1(z) + \lambda_R\left[k + 1 - \frac{m^*\omega z}{\hbar} + z\frac{\mathrm{d}}{\mathrm{d}z}\right]\phi_2(z) = 0 \\ &\hbar\omega\left[2z\frac{\mathrm{d}}{\mathrm{d}z} + k + 2 + \frac{(k+1)\omega_c}{2\omega} + \frac{\mu gB}{2\hbar\omega} - \frac{E}{\hbar\omega}\right]\phi_2(z) + \lambda_R\left[\frac{m^*\omega}{\hbar} - \frac{\mathrm{d}}{\mathrm{d}z}\right]\phi_1(z) = 0 \end{aligned} \tag{21}$$

where $z = xy$ and E is the eigenvalues of the Hamiltonian H and $\phi_1(z)$ and $\phi_2(z)$ correspond up and down eigenstates of the Hamiltonian H, respectively. Following Reik's analysis [19] which was constructed to obtain the solution of the $E \times \varepsilon$ Jahn–Teller Hamiltonian, one can obtain the isolated exact solution of the differential equations. Here, we present a solution of the problem in the framework of the quasi-exactly solvable problem.

4. Solution of the Hamiltonian

In the previous section, we have formulated the Rashba Hamiltonian based on the two-boson operators and we have discussed its transformation to the one variable differential equation. In this section, we present a transformation procedure which leads to the quasi-exact solution of the Hamiltonian (7). The Hamiltonian has been characterized by the two-boson operator. It is possible to transform the Hamiltonian that can be characterized by a one-boson operator. The connection between two- and single-boson Hamiltonian is given by a similarity transformation induced by the metric [28, 29]:

$$S = \left(a_2^+\right)^{a_1^+ a_1 + \sigma_+\sigma_-} \equiv \left(\sigma_-\sigma_+ + \sigma_+\sigma_- a_2^+\right)\left(a_2^+\right)^{a_1^+ a_1}. \tag{22}$$

The operator $\left(a_2^+\right)^{a_1^+ a_1}$ acts on the state $|n_1, n_2\rangle$ as follows:

$$\left(a_2^+\right)^{a_1^+ a_1}|n_1, n_2\rangle = \left(a_2^+\right)^{n_1}|n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2+n_1)!}}\,|n_1, n_2+n_1\rangle \tag{23}$$

then we can easily obtain the action of the operator S on the two-component state

$$S\left(|n_1, n_2\rangle\,|\uparrow\rangle + |n_1, n_2\rangle\,|\downarrow\rangle\right) = \sqrt{\frac{(n_2+n_1)!}{n_2!}} \times \left(\sqrt{(n_2+n_1+1)}\,|n_1, n_2+n_1+1\rangle\,|\uparrow\rangle + |n_1, n_2+n_1\rangle\,|\downarrow\rangle\right). \tag{24}$$

Since a_1 and a_2 commute, the transformation of a_1 and a_1^+ under S can be obtained by writing $a_2^+ = e^b$, with $[a_1, b] = \left[a_1^+, b\right] = 0$,

$$S^{-1}a_1 S = a_1 a_2^+; \qquad S^{-1}a_1^+ S = a_1^+\left(a_2^+\right)^{-1} \tag{25}$$

and transformation of a_2 and a_2^+ as follows:

$$S^{-1}a_2 S = a_2 + \left(a_1^+ a_1 + \sigma_+\sigma_-\right)\left(a_2^+\right)^{-1}; \qquad S^{-1}a_2^+ S = a_2^+. \tag{26}$$

Similarly, the transformation of the Pauli matrices are

$$S^{-1}\sigma_+ S = \sigma_+\left(a_2^+\right)^{-1}; \qquad S^{-1}\sigma_- S = \sigma_- a_2^+; \qquad S^{-1}\sigma_0 S = \sigma_0. \tag{27}$$

The Hamiltonian H can be transformed as

$$\widetilde{H} = S^{-1}HS = \hbar\omega\left(2a_1^+ a_1 + a_2^+ a_2 + \sigma_+\sigma_- + 1\right) - \frac{\hbar\omega_c}{2}\left(a_2^+ a_2 + \sigma_+\sigma_-\right) - \kappa\left[(1-a_1)\sigma_+ + \left(a_2 a_2^+ + a_1^+ a_1 + \sigma_+\sigma_- - a_1^+\right)\sigma_-\right] + \frac{1}{2}g\mu B\sigma_0. \tag{28}$$

It is obvious that $\widetilde{H}$ can be characterized by a fixed number $a_2^+ a_2 = -j - 1$. Here, j takes integer values. The transformed Hamiltonian $\widetilde{H}$ includes a one-boson operator and possessing infinitely many finite-dimensional invariant subspaces with a basis function:

$$\phi_{n,n+1}(z) = \begin{pmatrix} p_n(z) \\ q_{n+1}(z) \end{pmatrix}, \tag{29}$$

where $\phi(z)$ is two-component spinor and $p_n(z)$ and $q_{n+1}(z)$ are polynomials of degree n and $n+1$, respectively. Substituting Bargmann–Fock space realizations of the bosonic operators (17) by changing variable $z = \sqrt{\frac{m^*\omega}{\hbar}}x$, we obtain the following single-variable differential equation:

$$\widetilde{H} = \hbar\omega\left(2z\frac{\mathrm{d}}{\mathrm{d}z} - j + \sigma_+\sigma_-\right) + \frac{\hbar\omega_c}{2}\left(j + 1 - \sigma_+\sigma_-\right) + g\mu B\frac{\sigma_0}{2} - \kappa\left[\left(1 - \frac{\mathrm{d}}{\mathrm{d}z}\right)\sigma_+ + \sigma_-\left(z\frac{\mathrm{d}}{\mathrm{d}z} - j - z\right)\right]. \tag{30}$$

The eigenvalue problem can be expressed as

$$\widetilde{H}\phi(z) = E\phi(z). \tag{31}$$

The action of the $\widetilde{H}$ on the basis function $\phi(z)$ gives the following recurrence relation:

$$\begin{aligned} &[2n\hbar\omega - E + \epsilon_j + \epsilon_b]p_n(E) - \kappa\,[q_{n+1}(E) - (n+1)q_n(E)] = 0 \\ &[2n\hbar\omega - E + \epsilon_j - \epsilon_b]q_{n+1}(E) + \kappa\,[p_{n+1}(E) + (j-n)p_n(E)] = 0 \end{aligned} \tag{32}$$

where $p_n(E)$ and $q_n(E)$ are the coefficients of the polynomials $p_n(z)$ and $q_n(z)$, respectively. The justified parameters are given by

$$\epsilon_j = \hbar\omega\left(\frac{1}{2} - j + \frac{(3+2j)\omega_c}{4\omega}\right), \qquad \epsilon_b = \left(\frac{\hbar\omega_c}{4} - \frac{\mu g B}{2} + \frac{\hbar\omega}{2}\right). \tag{33}$$

It is necessary that the determinant of these sets be equal to zero giving the compatibility conditions that establish the locations of the exact eigenvalues on the energy baseline. The recurrence relation implies that the wavefunction is itself the generating function of the energy polynomials. If E_k is the roots of the (32), the eigenfunction truncated at $n = j$ and it is the exact eigenvalues of the Hamiltonian $\widetilde{H}$. These recurrence relations can be written in the matrix form:

$$\begin{bmatrix} E_+ & -\kappa & 0 & 0 & 0 & \cdot \\ j\kappa & E_- & \kappa & 0 & 0 & \cdot \\ 0 & 2\kappa & E_+ + 2\hbar\omega & -\kappa & 0 & \cdot \\ 0 & 0 & (j-1)\kappa & E_- + 2\hbar\omega & \kappa & \cdot \\ 0 & 0 & 0 & 3\kappa & E_+ + 3\hbar\omega & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \end{bmatrix} \begin{bmatrix} p_0 \\ q_1 \\ p_1 \\ q_2 \\ p_2 \\ \vdots \end{bmatrix} = 0 \tag{34}$$

where $E_\pm = \epsilon_j \pm \epsilon_b - E$. It is obvious that the determinant of the matrix forms polynomials in E and the first few of them are given by

$$\begin{aligned} D_0(E) &= E_+E_- \\ D_1(E) &= (E_- + 2\hbar\omega)[E_+E_-(E_+ + 2\hbar\omega) - \kappa^2(E_+ - 2\hbar\omega)] \\ D_2(E) &= (E_- + 4\hbar\omega)\big[E_+E_-(E_+ + 2\hbar\omega)(E_- + 2\hbar\omega)(E_+ + 4\hbar\omega) \\ &\quad + \kappa^2\big(2\hbar\omega E_+(E_- + 4\hbar\omega) + 162\hbar^2\omega^2(E_- + 2\hbar\omega) - 2E_+^2E_-\big) + 2\kappa^4(E_+ - 2\hbar\omega)\big] \end{aligned} \tag{35}$$

for the values $j = 0$, 1 and 2, respectively. The eigenvalues E of the Hamiltonian (7) can be determined by finding the roots of the polynomials (35).

5. Conclusion

We presented a QES to the problem of an electron in a quantum dot in the presence of both the magnetic field and spin–orbit coupling. Our formulation gives a unified treatment of the solution of the spin-boson physical systems. It has been shown that the mathematical structures of the quantum dot Hamiltonian including Rashba spin–orbit interaction and some of the spin-boson physical systems are identical. We have also shown that the Hamiltonian possesses $osp(2, 2)$ hidden symmetries. The suggested approach can be modified to solve the quantum dot Hamiltonian including Dresselhaus interaction.

Furthermore, we have presented a transformation procedure that offers several advantages, especially if one wishes to describe the eigenvalues of the Hamiltonian. It is obvious that the algebraic techniques have been used in a variety of problems to compute their spectra. We

have presented the steps towards an extension of the algebraic formulation of the quantum dot Hamiltonians.

The technique given in this paper can be extended in several ways. The Hamiltonian of a quantum dot including position-dependent effective mass may be formulated and solved within the procedure given here. We hope that our method leads to interesting results on the spin–orbit effects in quantum dots in future research.

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